Indian Statistical Institute
II Sem, Mid-Semestral Examination 2008-2009
M.Math. II year
Partial Deferential Equations

Date:02-03-2009 Duration: 3 Hours Max Marks 40

In Part-A you can get maximum of 15 and in part-B you can get maximum of 25.

Part-A

1. Let $h_n \varepsilon R, h_n \longrightarrow 0$. For $\phi$ in $C^\infty_{cpt}(R)$ define $\phi_n(x) = \frac{\phi[x+h_n]-\phi(x)}{h_n}$. Show that $\phi_n \longrightarrow \phi^1$ in the topology for $C^\infty_{cpt}(R)$. [2]

2. Let $T$ be a distribution on $R$, define $T_h\phi = T\phi_{-h}$ where $\phi_h(x) = \phi(x+h)$ for $\phi$ in $C^\infty_{cpt}(R)$. Prove $\frac{T_n-I}{h} \longrightarrow T^1$ as $h \longrightarrow 0$. [1]

3. Let $f : R \longrightarrow R$ be given by $f = 0$ on $[-\infty, 0] \cup [1, \infty], f(x) = 2$ for $0 < x < 1, x$ irrational and $f(x) = \frac{1}{x}$ for $0 < x < 1, x$ rational. Define $T\phi = \int f\phi$ for $\phi$ in $C^\infty_{cpt}(R)$. Find the distributional derivative of $T$. [2]

4. (a) For a distribution $T$ on $R$ and a $C^\infty$ function $\psi$ on $R$ define $\psi T$ show that $(\psi T)^1 = \psi^1 T + \psi T^1$. [2]

   (b) For $\psi_1, \psi_2$ $m^1$ $C^\infty(R)$ and $T$ as above prove that $\psi_1((\psi_2\ T) = (\psi_1\ \psi_2)\ T$. [1]

5. Let $f : R \longrightarrow R$, be given by $f(x) = \frac{1}{x}$. For $\phi \varepsilon C^\infty_{cpt}(R)$, define $T\phi = CPV \int f\phi$, where $CPV$ is cauchy principal value. Show that $T$ is a distribution. [2]

6. Let $S : C^1[0, 1] \longrightarrow \mathbb{C}$ be any linear continuous map. Show that there exists a complex measure $\alpha$ on $[0, 1]$ such that $S(f) = \int f^1 d\ \alpha$ for all $f$ in$C^\infty_{cpt}(0, 1)$. [4]

7. (a) What is the meaning of distribution $T = f$ on an open set $G$ for some $f \varepsilon L^1_{loc}(G)$. [Here $T$ is distribution on $R$ and $G$ open in $R$] [1]

   (b) For any $\psi$ in $C^\infty(R)$, show that $\psi T = \psi f$ on $G$ where $G, T, f$ are as above. [1]

Part-B

8. Let $P$ be an elliptic operator on $R^d$, $u$ is a distribution on $R^d \psi_o, \psi_1 \varepsilon C_{cpt}^{\infty}(R^d)$ with $\psi_o\psi_1 = \psi_1$ Let $P(\partial_x)u \varepsilon C^{\infty}(R^d)$, if $\psi_0 u \varepsilon H^{s_o}(R^d)$. Show that $\psi_1 u \varepsilon H^{s_o+1}(R^d)$. [5]

9. (a) Let $P : R^d \longrightarrow C$ be any polynomial of degree m. If there are constants $C_0 > 0, C_1$ such that $C_o(1 + 1\xi 1)^m \leq [P(\xi)] + C_1$ then show that $P$ is an elliptic polynomial. [2]

   (b) Let $P$ be any elliptic operator on $R^d$ : $f$ is any polynomial, in one variable show that $f(P(\partial_x))$ is also an elliptic operator on $R^d$. [2]

10. (a) If $P : R^d \longrightarrow C$ is any polynomial and bounded show that $P$ is a constant. [1]

    (b) Let $P(\partial)$ be any $PDO$ (Partial Differential Operator) on $R^d$ with constant coefficients. Let $P(i\xi) \neq o$ for $\xi \neq o$. If U is a tempered distribution such that $P(\partial)U = o$, then $U$ is a polynomial. [2]

    (c) Let $P$ be as in $(b)$. If $f \varepsilon C^{\infty}(R^d)$ is bounded and $P(\partial)f = 0$, then $f$ is constant. [1]

11. Let $P(\partial)$ be any $PDO$ on $R^d$ of order $m$ with constant coefficients such that $P(i\xi) = \xi_1^{m-1}Q_0(\xi^1) + .... + \xi_1^0 Q_m(\xi^1)$ for $(\xi_1, \xi^1)\varepsilon R \times R^{d-1} = R^d$. Show that $P(\partial)$ has a fundamental solution. [10]

12. Find a fundamental solution for the operator $\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$ on $R^2$ and prove your claim. [4]

13. Let $f_1(x, y) = \frac{x}{x^2+y^2}, f_2(x, y)= \log(x^2 + y^2)$ on $R^2 - \{0\}$ show that $f_1, f_2$ are real analytic on $R^2 - \{0\}$. [4]

14. Let $A : R^d \longrightarrow R^d$ be a linear map satisfying $A^1A = I = AA^1$. Let $f : R^d - \{0\} \longrightarrow R$ satisfy $\triangle f = 0$. Define $g : R^d - \{0\} \longrightarrow R$ by $g(x) = f(Ax)$. Show that $\triangle g = 0$. [3]